Analytic number theory
Solutions to Exercise Sheet 3

Exercise 1. We use the Möbius function to detect the coprimality condition, as we did in Exercise Sheet 1.

$$\begin{aligned}\sum_{\substack{n\le x\\(n,q)=1}}\frac{1}{n}&=\sum_{n\le x}\sum_{d\mid(n,q)}\frac{\mu(d)}{n}\\&=\sum_{\substack{d\le x\\d\mid q}}\sum_{a\le x/d}\mu(d)\frac{1}{ad}\\&=\sum_{\substack{d\le x\\d\mid q}}\frac{\mu(d)}{d}\sum_{a\le x/d}\frac{1}{a}.\end{aligned}$$

Now recall that

$$\sum_{a\le x}\frac{1}{a}=\log x+\gamma+O(1/x). \tag{0.1}$$

We plug-in (0.1) and get

$$\begin{aligned}\sum_{\substack{n\le x\\(n,q)=1}}\frac{1}{n}&=\sum_{d\mid q}\frac{\mu(d)}{d}(\log(x/d)+\gamma+O(d/x))\\&=\log x\sum_{d\mid q}\frac{\mu(d)}{d}+\sum_{d\mid q}\frac{\mu(d)}{d}(\gamma-\log d)+O\left(\frac{\tau(q)}{x}\right).\end{aligned}$$

Using $\sum_{d\mid q}\frac{\mu(d)}{d}=\frac{\varphi(q)}{q}$ we obtain the claim with $C(q)=\sum_{d\mid q}\frac{\mu(d)}{d}(\gamma-\log d)$.

Exercise 2. First we recall the sum

$$\begin{aligned}S:&=\sum_{n\le XY}(f*g)(n)\\&=\sum_{n\le XY}\sum_{ab=n}f(a)g(b).\end{aligned}$$

Then by the Dirichlet's hyperbola method, we can divide the range $U=\{n:n\le XY\}$ in two regions $A:=\{(a,b):a\le X\}$ and $B:=\{(a,b):b\le Y\}$. Now as $U\subset A\cup B$ so we get that

$$|U|=|U\cap A|+|U\cap B|-|U\cap A\cap B|.$$

Using this we can write the sum S as

$$S=\sum_{a\le X}f(a)\sum_{b\le XY/a}g(b)+\sum_{b\le Y}g(b)\sum_{a\le XY/b}f(a)-\left(\sum_{a\le X}f(a)\right)\left(\sum_{b\le Y}g(b)\right),$$

giving the desired result.

Exercise 3. Here we will try to use the previous exercise. Consider the functions $f(n) = n$ and $g(n) = 1$ for all $n \in \mathbb{N}$. Then we see that

$$\sigma(n) = f * g(n).$$

Then taking $X = 1, Y = x$ and using the previous exercise (or doing directly) we get that

$$\begin{aligned}\sum_{n\le x}\sigma(n) &= \sum_{b\le x}\sum_{a\le x/b} a \\ &= x^2\sum_{b\le x}\frac{1}{2b^2} + \mathrm{O}(\sum_{b\le x}\frac{x}{b}) \\ &= x^2\sum_{b\le x}\frac{1}{2b^2} + \mathrm{O}(x\log x).\end{aligned} \tag{0.2}$$

Now for the first sum we have

$$\begin{aligned}\sum_{b\le x}\frac{1}{b^2} &= \sum_{b=1}^{\infty}\frac{1}{b^2} - \sum_{b>x}\frac{1}{b^2} \\ &= \zeta(2) + \mathrm{O}\left(\int_x^\infty \frac{1}{u^2}\mathrm{d}u\right) \\ &= \zeta(2) + \mathrm{O}(x^{-1}).\end{aligned}$$

Putting this in the equation (0.2) we get our desired result.

Exercise 4. (a) Let $n = p_1^{l_1}\cdots p_k^{l_k}$ for pairwise distinct primes $p_1, \ldots, p_k$ and non-negative integers $l_1, \ldots, l_k$:

$$\begin{aligned}\epsilon * \mu^2(n) &= \sum_{d|n}\mu^2(d) \\ &= \sum_{\substack{d|n \\ d \text{ square-free}}} 1 \\ &= \sum_{j_1=0}^{1}\cdots\sum_{j_k=0}^{1} 1 = 2^k = 2^{\omega(n)}.\end{aligned}$$

(b) Let $F(x) = \sum_{n\le x}\mu(n)^2$. By Exercise Sheet 2, Exercise 2, we have $F(x) = c_0x + E(x)$, with $E(x) = O(\sqrt{x})$. By Abel summation:

$$\begin{aligned}\sum_{n\le x}\frac{\mu(n)^2}{n} &= 1 + \sum_{1<n\le x}\frac{\mu(n)^2}{n} \\ &= 1 + \frac{F(x)}{x} - \frac{F(1)}{1} + \int_1^x \frac{F(t)}{t^2}\,\mathrm{d}t \\ &= c_0 + \frac{E(x)}{x} + c_0\int_1^x \frac{1}{t}\,\mathrm{d}t + \int_1^x \frac{E(t)}{t^2}\,\mathrm{d}t \\ &= c_0 + \frac{E(x)}{x} + c_0\log x + c - \int_x^\infty \frac{E(t)}{t^2}\,\mathrm{d}t,\end{aligned}$$

where $c = \int_1^\infty \frac{E(t)}{t^2}\,\mathrm{d}t < \infty$. Note that since $E(t) = O(\sqrt{t})$, then both $\frac{E(x)}{x}$ and $-\int_x^\infty \frac{E(t)}{t^2}\,\mathrm{d}t$ are $O(1/\sqrt{x})$. Conclude by setting $c_1 = c_0 + c$.

(c) We have

$$\begin{aligned}\sum_{n\le x} 2^{\omega(n)} &= \sum_{n\le x}\sum_{d|n}\mu(d)^2\\ &= \sum_{n\le x}\sum_{da=n}\mu(d)^2\\ &\overset{\dagger}{=} \sum_{d\le\sqrt{x}}\mu(d)^2\sum_{a\le x/d}1+\sum_{a\le\sqrt{x}}\sum_{d\le x/a}\mu(d)^2-\sum_{a,d\le\sqrt{x}}\mu(d)^2,\end{aligned}$$

where in † we used the hyperbola method. We estimate each term individually, for the first one using the previous subexercise we get:

$$\sum_{d\le\sqrt{x}}\mu(d)^2\sum_{a\le x/d}1 = x\sum_{d\le\sqrt{x}}\frac{\mu(d)^2}{d}+O(\sqrt{x}) = \frac{c_0}{2}x\log x+c_1x+O(\sqrt{x}).$$

For the second:

$$\begin{aligned}\sum_{a\le\sqrt{x}}\sum_{d\le x/a}\mu(d)^2 &= \sum_{a\le\sqrt{x}} c_0\frac{x}{a}+E(x/a)\\ &= c_0x\sum_{a\le\sqrt{x}}\frac{1}{a}+\sum_{a\le\sqrt{x}}E(x/a)\\ &= \frac{c_0}{2}x\log x+c_0\gamma x+O(\sqrt{x})+\sum_{a\le\sqrt{x}}E(x/a).\end{aligned}$$

We estimate

$$\begin{aligned}\sum_{a\le\sqrt{x}}E(x/a) &\ll \sqrt{x}\sum_{a\le\sqrt{x}}\frac{1}{\sqrt{a}}\\ &\le \sqrt{x}(x^{1/4}+1)=O(x^{3/4}).\end{aligned}$$

The last term can be estimated

$$\begin{aligned}\sum_{d\le\sqrt{x}}\mu(d)^2\sum_{a\le\sqrt{x}} &= \sqrt{x}\sum_{d\le\sqrt{x}}\mu(d)^2+O(\sum_{d\le\sqrt{x}}\mu(d)^2)\\ &= c_0x+\sqrt{x}O(x^{1/4})+O(\sqrt{x})=c_0x+O(x^{3/4}).\end{aligned}$$

In particular by setting $c_2=c_0$, $c_3=c_0+c_0\gamma+c_1$ we obtain the desired result.

Exercise 5. (a) We follow the hint

$$\begin{aligned}1 &= \sum_{n\le x}e(n)\\ &= \sum_{n\le x}\sum_{d|n}\mu(d)\\ &= \sum_{d\le x}\mu(d)\sum_{\substack{n\le x\\ d|n}}1\\ &= \sum_{d\le x}\mu(d)\lfloor x/d\rfloor\\ &= x\sum_{d\le x}\frac{\mu(d)}{d}-\sum_{d\le x}\mu(d)\{x/d\}.\end{aligned}$$

If

$$|\sum_{d\le x}\mu(d)\{x/d\}| \le x-1, \tag{0.3}$$

then by triangle inequality

$$\left|\sum_{d\le x}\frac{\mu(d)}{d}\right| \le 1.$$

The inequality (0.3) follows by triangle inequality and the observation that the first two summands have opposite sign, hence the absolute value of their sum is ≤ 1 and one estimates the remaining terms via triangle inequality.

(b) We have

$$\Lambda(n)=\sum_{d|n}\mu(d)\log(n/d)=\log(n)\sum_{d|n}\mu(d)-\sum_{d|n}\mu(d)\log(d).$$

The first summand is always 0: if $n=1$ the term $\log(n)$ vanishes, else we have $\sum_{d|n}\mu(d)=0$.

(c) By the previous point we have $\Lambda=-\mu\cdot\log *\epsilon$. Hence

$$\mu(n)\log(n)=-\Lambda*\mu(n)=-\sum_{d|n}\Lambda(n/d)\mu(d).$$

Hence

$$\sum_{n\le x}\mu(n)\log(n)=-\sum_{d\le x}\mu(d)\sum_{\substack{n\le x\\ d|n}}\Lambda\left(\frac{n}{d}\right)=-\sum_{d\le x}\mu(d)\psi(x/d).$$

(d) Recall that $\psi(x)\sim x$ means that for fixed $\epsilon>0$ there exists $x_0>0$ so that for all $x\ge x_0$ it holds that

$$|\frac{\psi(x)}{x}-1|<\epsilon.$$

Equivalently $|\psi(x)-x|<\epsilon x$. Hence

$$\begin{aligned}\sum_{d\le\frac{x}{x_0}}\left|\psi\left(\frac{x}{d}\right)-\frac{x}{d}\right| &\le \epsilon x\sum_{d\le\frac{x}{x_0}}\frac{1}{d}\\ &= x\epsilon(\log(x/x_0)+O(1))=(x\log x)\epsilon+O(x).\end{aligned}$$

We proceed in estimating $|H(x)|$: Fix $\epsilon>0$ and let x_0 as abpve. We can rewrite

$$H(x)=-\sum_{d\le x/x_0}\mu(d)(\psi(x/d)-x/d))-x\sum_{d\le x/x_0}\frac{\mu(d)}{d}-\sum_{x/x_0<d\le x}\mu(d)\psi(x/d).$$

The second summand is estimated by bullet (a). The third summand is estimated by

$$|\sum_{x/x_0<d\le x}\mu(d)\psi(x/d)|\le\sum_{x/x_0<d\le x}x/d\log(x/d)\le x\cdot x_0\log(x_0),$$

where in the first inequality we used the triangle inequality and the trivial bound $|\psi(x)|\le x\log x$. In particular we get

$$|H(x)|=(x\log x)\epsilon+O(x).$$

(e) Abel summation yields

$$\sum_{n\le x}\mu(n) = 1 + \sum_{2\le n\le x}\frac{\mu(n)\log(n)}{\log(n)}$$
$$= 1 + \frac{H(x)}{\log x} - \frac{H(3/2)}{\log(3/2)} + \int_{3/2}^{x}\frac{H(t)}{t\log^2(t)}\,\mathrm{d}t.$$

We estimate the integral by

$$\left|\int_{3/2}^{x}\frac{H(t)}{t\log^2 t}\,\mathrm{d}t\right| \le \int_{3/2}^{x}\frac{|H(t)|}{t\log^2 t}\,\mathrm{d}t$$
$$= \epsilon\int_{3/2}^{x}\frac{1}{\log t}\,\mathrm{d}t + O\Big(\int_{3/2}^{x}\frac{1}{\log^2 t}\,\mathrm{d}t\Big)$$
$$= O(x/\log x),$$

where we used the previous subexercise and the estimation for $Li(x)$, see exercise sheet 1 for the latter. In particular we have

$$\sum_{n\le x}\mu(n) = 1 + \epsilon(x\log x) + O(x/\log x),$$

after dividing by x we get the desired result.